

## DOUBLE-NORMAL PAIRS IN SPACE

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**Abstract.** A *double-normal pair* of a finite set  $S$  of points that spans  $\mathbb{R}^d$  is a pair of points  $\{p, q\}$  from  $S$  such that  $S$  lies in the closed strip bounded by the hyperplanes through  $p$  and  $q$  perpendicular to  $pq$ . A double-normal pair  $\{p, q\}$  is *strict* if  $S \setminus \{p, q\}$  lies in the open strip. The problem of estimating the maximum number  $N_d(n)$  of double-normal pairs in a set of  $n$  points in  $\mathbb{R}^d$ , was initiated by Martini and Soltan [*Discrete Math.* **290** (2005), 221–228]. It was shown in a companion paper that in the plane, this maximum is  $3\lfloor n/2 \rfloor$ , for every  $n > 2$ . For  $d \geq 3$ , it follows from the Erdős–Stone theorem in extremal graph theory that  $N_d(n) = \frac{1}{2}(1 - 1/k)n^2 + o(n^2)$  for a suitable positive integer  $k = k(d)$ . Here we prove that  $k(3) = 2$  and, in general,  $\lceil d/2 \rceil \leq k(d) \leq d - 1$ . Moreover, asymptotically we have  $\lim_{n \rightarrow \infty} k(d)/d = 1$ . The same bounds hold for the maximum number of strict double-normal pairs.

**§1. Introduction.** Let  $V$  be a set of  $n$  points that spans  $\mathbb{R}^d$ . A *double-normal pair* of  $V$  is a pair of distinct points  $\{p, q\}$  in  $V$  such that  $V$  lies in the closed strip bounded by the hyperplanes  $H_p$  and  $H_q$  through  $p$  and  $q$ , respectively, that are perpendicular to the line  $pq$ . A double-normal pair  $\{p, q\}$  is *strict* if  $V \setminus \{p, q\}$  is disjoint from the hyperplanes  $H_p$  and  $H_q$ . Define the *double-normal graph* of  $V$  as the graph on the vertex set  $V$  in which two vertices  $p$  and  $q$  are joined by an edge if and only if  $\{p, q\}$  is a double-normal pair. The number of edges of this graph, that is, the number of double-normal pairs induced by  $V$  is denoted by  $N(V)$ .

We define the *strict double-normal graph* of  $V$  analogously and denote its number of edges by  $N'(V)$ .

Martini and Soltan [10, Problems 3 and 4] asked for the maximum numbers  $N_d(n)$  and  $N'_d(n)$  of double-normal pairs and strict double-normal pairs of a set of  $n$  points in  $\mathbb{R}^d$ :

$$N_d(n) := \max_{\substack{V \subset \mathbb{R}^d \\ |V|=n}} N(V)$$

and

$$N'_d(n) := \max_{\substack{V \subset \mathbb{R}^d \\ |V|=n}} N'(V).$$

Received 2 April 2014, published online 14 August 2014.

MSC (2010): 52C10 (primary).

Part of this research was performed while the authors were visiting the Institute for Pure and Applied Mathematics (IPAM), which is supported by the National Science Foundation.

<sup>†</sup> Research partially supported by Swiss National Science Foundation grants 200021-137574 and 200020-144531, by Hungarian Science Foundation grant OTKA NN 102029 under the EuroGIGA programs ComPoSe and GraDR, and by NSF grant CCF-08-30272.

Clearly, we have  $N(V) \geq N'(V)$  and  $N_d(n) \geq N'_d(n)$ . It is not difficult to see that  $N'_2(n) = n$ . In another paper [12] we show that  $N_2(n) = 3\lfloor n/2 \rfloor$ . Here we only consider the case  $d \geq 3$ .

**THEOREM 1.** *The maximum number of double-normal and strict double-normal pairs in a set of  $n$  points in  $\mathbb{R}^3$  satisfy  $N_3(n) = n^2/4 + o(n^2)$  and  $N'_3(n) = n^2/4 + o(n^2)$ .*

In fact, since the collection of double-normal graphs in Euclidean space is closed under the taking of induced subgraphs, the Erdős–Stone theorem [3] implies that for each  $d \in \mathbb{N}$ , there exist unique  $k(d), k'(d) \in \mathbb{N}$  such that  $N_d(n) = \frac{1}{2}(1 - 1/k(d))n^2 + o(n^2)$  and  $N'_d(n) = \frac{1}{2}(1 - 1/k'(d))n^2 + o(n^2)$ . The number  $k(d)$  (respectively  $k'(d)$ ) can also be characterized as the largest  $k$  such that complete  $k$ -partite graphs with arbitrarily many points in each class occur as subgraphs of double-normal (respectively strictly double-normal) graphs in  $\mathbb{R}^d$ . Theorem 1 states that  $k(3) = k'(3) = 2$  and is a special case of the next theorem.

**THEOREM 2.** *For each  $d$ , there exist unique integers  $k(d), k'(d) \geq 1$  such that  $N_d(n)$ , the maximum number of double-normal pairs, and  $N'_d(n)$ , the maximum number of strict double-normal pairs in a set of  $n$  points in  $\mathbb{R}^d$ , satisfy*

$$N_d(n) = \frac{1}{2} \left( 1 - \frac{1}{k(d)} \right) n^2 + o(n^2)$$

and

$$N'_d(n) = \frac{1}{2} \left( 1 - \frac{1}{k'(d)} \right) n^2 + o(n^2).$$

For any  $d \geq 3$ , we have

$$\lceil d/2 \rceil \leq k'(d) \leq k(d) \leq d - 1.$$

Asymptotically, as  $d \rightarrow \infty$ , we have

$$k(d) \geq k'(d) \geq d - O(\log d).$$

Although this theorem gives the exact values  $k(3) = k'(3) = 2$ , we do not know whether  $k(4)$  or  $k'(4)$  equals 2 or 3.

Two notions related to double-normal pairs have been studied before. We define a *diameter pair* of  $S$  to be a pair of points  $\{p, q\}$  in  $S$  such that  $|pq| = \text{diam}(S)$ . Note that a diameter pair is also a strictly double-normal pair. The maximum number of diameter pairs in a set of  $n$  points is known for all  $d \geq 2$ , and in the case of  $d \geq 4$ , if  $n$  is sufficiently large [1, 4–6, 13, 14]. We call a pair  $pq$  of a set  $S \subset \mathbb{R}^d$  *antipodal* if there exist parallel hyperplanes  $H_1$  and  $H_2$  through  $p$  and  $q$ , respectively, such that  $S$  lies in the closed strip bounded by the hyperplanes. The pair is called *strictly antipodal* if there exist parallel hyperplanes through  $p$  and  $q$  such that  $S \setminus \{p, q\}$  lies in the open strip bounded by the hyperplanes. Clearly, a (strictly) double-normal pair of a set is also a

(strictly) antipodal pair. The problem of determining the asymptotic behaviour of the maximum number of antipodal or strictly antipodal pairs in a set of  $n$  points is open already in  $\mathbb{R}^3$ . For a thorough discussion of antipodal pairs, see the series of papers [7–9].

The paper is structured as follows. In §2, we collect some geometric lemmas on double-normal pairs. They are applied in §3 together with a Ramsey-type argument to derive the upper bound of Theorem 2 (Theorem 7). Finally, in §4 we show the two lower bounds of Theorem 2 (Corollaries 10 and 16). The asymptotic lower bound follows from a random construction closely related to the construction by Erdős and Füredi [2] of strictly antipodal sets of size exponential in the dimension.

We use the following notation. The inner product of  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$  is denoted by  $\langle \mathbf{x}, \mathbf{y} \rangle$ , the linear span of  $S \subset \mathbb{R}^d$  by  $\text{lin } S$ , the convex hull of  $S$  by  $\text{conv } S$ , the diameter of  $S$  by  $\text{diam}(S)$ , the cardinality of a finite set  $S$  by  $|S|$ , and the complete  $k$ -partite graph with  $N$  vertices in each class by  $K_k(N)$ . An angle with vertex  $\mathbf{b}$  and sides  $\mathbf{ba}$  and  $\mathbf{bc}$  is denoted by  $\angle \mathbf{abc}$ , which we also use to denote its angular measure. All angles in this paper have angular measure in the range  $(0, \pi)$ . The Euclidean distance between  $\mathbf{p}$  and  $\mathbf{q}$  is denoted  $|\mathbf{pq}|$ .

§2. *Geometric properties of the double-normal relation.* Here we collect some elementary geometric properties of double-normal pairs. They will be used in the next section where we find upper bounds to  $k(d)$ .

If a unit vector  $\mathbf{u}$  is almost orthogonal to two given unit vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , then  $\mathbf{u}$  is still almost orthogonal to any unit vector in the span of  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , with an error that becomes worse the closer  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are to each other. The next lemma quantifies this observation.

LEMMA 3. *Let  $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2$  be unit vectors with  $\mathbf{u}_1 \neq \pm \mathbf{u}_2$ , such that for some  $\varepsilon_1, \varepsilon_2 > 0$ ,  $|\langle \mathbf{u}, \mathbf{u}_1 \rangle| \leq \varepsilon_1$  and  $|\langle \mathbf{u}, \mathbf{u}_2 \rangle| \leq \varepsilon_2$ . Then for any unit vector  $\mathbf{v} \in \text{lin}\{\mathbf{u}_1, \mathbf{u}_2\}$  we have  $|\langle \mathbf{u}, \mathbf{v} \rangle| < (\varepsilon_1 + \varepsilon_2)/\sin \theta$ , where  $\theta \in (0, \pi)$  satisfies  $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \cos \theta$ .*

*Proof.* Let  $\mathbf{u}'$  be the orthogonal projection of  $\mathbf{u}$  onto the plane  $\text{lin}\{\mathbf{u}_1, \mathbf{u}_2\}$ . Then the quantity  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}', \mathbf{v} \rangle$  is maximized when  $\mathbf{v}$  is a positive multiple of  $\mathbf{u}'$ , and then  $|\langle \mathbf{u}, \mathbf{v} \rangle| = |\mathbf{ou}'|$ . It follows from the hypotheses that  $\mathbf{u}'$  lies in the parallelogram  $P$  symmetric around  $\mathbf{o}$  with sides perpendicular to  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , respectively, and with the sides perpendicular to  $\mathbf{u}_i$  at distance  $2\varepsilon_i$ ,  $i = 1, 2$ . The sides of  $P$  form an angle of  $\theta$ , and their lengths are  $2\varepsilon_1/\sin \theta$  and  $2\varepsilon_2/\sin \theta$ . The maximum value of  $|\mathbf{ou}'|$  is attained at a vertex of the parallelogram  $P$ , that is,  $|\mathbf{ou}'|$  is at most half the largest diagonal of  $P$ . By the law of cosines, half a diagonal of  $P$  has length

$$\begin{aligned} & \sqrt{\frac{\varepsilon_1^2}{\sin^2 \theta} + \frac{\varepsilon_2^2}{\sin^2 \theta} \pm 2 \frac{\varepsilon_1 \varepsilon_2}{\sin^2 \theta} \cos \theta} \\ & < \sqrt{\frac{\varepsilon_1^2}{\sin^2 \theta} + \frac{\varepsilon_2^2}{\sin^2 \theta} + 2 \frac{\varepsilon_1 \varepsilon_2}{\sin^2 \theta}} = \frac{\varepsilon_1 + \varepsilon_2}{\sin \theta}. \end{aligned} \quad \square$$

Suppose that  $y_1, y_2, y_3$  are collinear, with  $y_2$  between  $y_1$  and  $y_3$ , and that  $x y_2$  is a double-normal pair in some set that contains  $x, y_1, y_2, y_3$ . Then, since the segment  $y_1 y_3$  has to lie in the half-space through  $y_2$  with normal  $y_2 x$ , it follows that  $y_1 y_3$  lies in the boundary of this half-space. That is,  $x y_2 \perp y_1 y_2$ . If  $y_1, y_2, y_3$  are close to collinear, then intuitively  $y_1 y_2$  will still be close to orthogonal to  $x y_2$ . This is the content of the next lemma.

LEMMA 4. *Let  $x, y_1, y_2, y_3$  be different points from  $V \subset \mathbb{R}^d$ , with  $x y_2$  a double-normal pair in  $V$ . Let  $\varepsilon > 0$  and suppose that  $\angle y_1 y_2 y_3 > \pi - \varepsilon$ . Let  $u$  be a unit vector parallel to  $y_1 y_2$  and  $v$  a unit vector parallel to  $x y_2$ . Then  $|\langle u, v \rangle| < \varepsilon$ .*

*Proof.* Without loss of generality,  $\varepsilon < \pi/2$ . Note that  $\angle x y_2 y_1, \angle x y_2 y_3 \leq \pi/2$ . Since also

$$\pi - \varepsilon < \angle y_1 y_2 y_3 \leq \angle y_1 y_2 x + \angle x y_2 y_3 \leq \angle y_1 y_2 x + \pi/2,$$

we obtain

$$\pi/2 - \varepsilon < \angle y_1 y_2 x \leq \pi/2,$$

and it follows that

$$|\langle u, v \rangle| = \cos \angle y_1 y_2 x < \cos(\pi/2 - \varepsilon) = \sin \varepsilon < \varepsilon. \quad \square$$

Consider the situation where  $y_1, y_2, y_3$  are “almost” collinear with  $y_2$  the “middle” point, but now there are two double-normal pairs  $x_1 y_2$  and  $x_2 y_2$  in a set that contains  $x_1, x_2, y_1, y_2, y_3$ . Then  $y_1, y_2, y_3$  all lie inside the wedge  $W$  formed by the intersection of the half-spaces  $H_1$  and  $H_2$  through  $y_2$  with normals  $x_1 - y_2$  and  $x_2 - y_2$ , respectively. If  $y_1, y_2, y_3$  are collinear with  $y_2$  between  $y_1$  and  $y_3$ , then necessarily  $y_1, y_2, y_3$  all lie on the “ridge”  $\text{bd } H_1 \cap \text{bd } H_2$  of the wedge  $W$ , and  $y_1 y_2$  is orthogonal to the plane  $\Pi$  through  $x_1, x_2, y_2$ . If  $y_1, y_2, y_3$  are close to collinear, then intuitively  $y_1 y_2$  will still be close to orthogonal to  $\Pi$ . The next lemma quantifies this intuition. It is an immediate consequence of Lemmas 3 and 4.

LEMMA 5. *Let  $x_1, x_2, y_1, y_2, y_3$  be different points in  $V \subset \mathbb{R}^d$ , with  $x_1 y_2$  and  $x_2 y_2$  double-normal pairs in  $V$ . Let  $\varepsilon > 0$ . Suppose that  $\angle y_1 y_2 y_3 > \pi - \varepsilon$ . Then for any unit vector  $u$  parallel to the line  $y_1 y_2$  and any unit vector  $v$  parallel to the plane  $x_1 x_2 y_2$  we have  $|\langle u, v \rangle| < 2\varepsilon/\sin \angle x_1 y_2 x_2$ .*

If the angle  $\angle x_1 y_2 x_2$  in the previous lemma is small, then the bound obtained may be too large to be useful. In the next lemma, we show that we can still obtain a small upper bound if  $|y_1 y_2|$  is much smaller than  $|x_1 x_2|$ . We need four double-normal pairs instead of the two required by Lemma 5, but we do not need  $y_3$ .

LEMMA 6. *Let  $x_i y_j, i, j = 1, 2$ , be four double-normal pairs in a set  $V \subset \mathbb{R}^d$  that contains  $x_1, x_2, y_1, y_2$ . Let  $u$  be a unit vector parallel to  $y_1 y_2$  and  $v$  a unit vector parallel to the plane  $x_1 x_2 y_2$ . Then*

$$|\langle u, v \rangle| \leq \frac{\sqrt{2}}{\cos^2 \angle x_1 y_2 x_2} \frac{|y_1 y_2|}{|x_1 x_2|}.$$

*Proof.* Let  $\mathbf{u} := |\mathbf{y}_1 \mathbf{y}_2|^{-1}(\mathbf{y}_1 - \mathbf{y}_2)$ ,  $\mathbf{u}_1 := |\mathbf{x}_1 \mathbf{y}_2|^{-1}(\mathbf{x}_1 - \mathbf{y}_2)$  and  $\mathbf{u}_2 := |\mathbf{x}_1 \mathbf{x}_2|^{-1}(\mathbf{x}_1 - \mathbf{x}_2)$ . Then  $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \cos \theta$  where  $\theta := \angle \mathbf{x}_2 \mathbf{x}_1 \mathbf{y}_2$ .

Since the angles  $\angle \mathbf{x}_1 \mathbf{y}_1 \mathbf{y}_2$ ,  $\angle \mathbf{x}_1 \mathbf{y}_2 \mathbf{y}_1$ ,  $\angle \mathbf{x}_2 \mathbf{y}_2 \mathbf{y}_1$  are non-obtuse, we obtain

$$\langle \mathbf{x}_1 - \mathbf{y}_1, \mathbf{y}_2 - \mathbf{y}_1 \rangle \geq 0, \quad (1)$$

$$\langle \mathbf{x}_1 - \mathbf{y}_2, \mathbf{y}_1 - \mathbf{y}_2 \rangle \geq 0, \quad (2)$$

and

$$\langle \mathbf{y}_2 - \mathbf{x}_2, \mathbf{y}_2 - \mathbf{y}_1 \rangle \geq 0. \quad (3)$$

From (1) we obtain  $\langle \mathbf{x}_1 - \mathbf{y}_2, \mathbf{y}_2 - \mathbf{y}_1 \rangle \geq -|\mathbf{y}_1 \mathbf{y}_2|^2$ , that is,

$$\langle \mathbf{u}, \mathbf{u}_1 \rangle \leq |\mathbf{y}_2 \mathbf{y}_1|/|\mathbf{x}_1 \mathbf{y}_2| =: \varepsilon_1.$$

From (2),  $\langle \mathbf{u}, \mathbf{u}_1 \rangle \geq 0$ . Next, add (1) and (3) to obtain  $\langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{y}_2 - \mathbf{y}_1 \rangle \leq |\mathbf{y}_1 \mathbf{y}_2|^2$ , that is,

$$\langle \mathbf{u}, \mathbf{u}_2 \rangle \leq |\mathbf{y}_1 \mathbf{y}_2|/|\mathbf{x}_1 \mathbf{x}_2| =: \varepsilon_2.$$

The analogues of (1) and (3) with  $\mathbf{x}_1$  and  $\mathbf{x}_2$  interchanged similarly give  $-\langle \mathbf{u}, \mathbf{u}_2 \rangle \leq \varepsilon_2$ . By Lemma 3, for any unit vector  $\mathbf{v}$  parallel to the plane  $\Pi$  through  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_2$ , that is, with  $\mathbf{v} \in \text{lin}\{\mathbf{u}_1, \mathbf{u}_2\}$ , we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \frac{\varepsilon_1 + \varepsilon_2}{\sin \theta}. \quad (4)$$

By the law of sines in  $\triangle \mathbf{x}_1 \mathbf{x}_2 \mathbf{y}_2$ ,

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{|\mathbf{x}_1 \mathbf{x}_2|}{|\mathbf{x}_1 \mathbf{y}_2|} = \frac{\sin \alpha}{\sin \varphi},$$

where  $\varphi := \angle \mathbf{x}_1 \mathbf{x}_2 \mathbf{y}_2$  and  $\alpha := \angle \mathbf{x}_1 \mathbf{y}_2 \mathbf{x}_2$ . It follows from (4) that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \frac{\varepsilon_2}{\sin \theta} \left( 1 + \frac{\sin \alpha}{\sin \varphi} \right).$$

Since  $\alpha, \theta, \varphi \leq \pi/2$  and  $\alpha + \theta + \varphi = \pi$ , we have

$$\sin \theta, \sin \varphi \geq \sin(\pi/2 - \alpha) = \cos \alpha,$$

hence

$$\begin{aligned} |\langle \mathbf{u}, \mathbf{v} \rangle| &\leq \frac{\varepsilon_2}{\cos \alpha} \left( 1 + \frac{\sin \alpha}{\cos \alpha} \right) = \frac{\varepsilon_2}{\cos^2 \alpha} (\cos \alpha + \sin \alpha) \\ &\leq \frac{\varepsilon_2}{\cos^2 \alpha} \sqrt{2} = \frac{\sqrt{2}}{\cos^2 \alpha} \frac{|\mathbf{y}_1 \mathbf{y}_2|}{|\mathbf{x}_1 \mathbf{x}_2|}. \end{aligned}$$

□

§3. *Upper bound on the number of double-normal pairs.* Recall that  $k(d)$  denotes the largest  $k$  such that for each  $N \in \mathbb{N}$ , the complete  $k$ -partite graph with  $N$  vertices in each class,  $K_k(N)$ , is a subgraph of some double-normal graph in  $\mathbb{R}^d$ .

THEOREM 7. *For all  $d \geq 3$ , we have  $k(d) \leq d - 1$ .*

This theorem is a straightforward consequence of the following technical result.

PROPOSITION 8. *There exist a family of  $k = k(d)$  not necessarily distinct points  $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$  and a family of  $k^2$  not necessarily distinct unit vectors  $\{\mathbf{u}_{i,j} : 1 \leq i, j \leq k\}$ , all in  $\mathbb{R}^d$ , such that the following hold:*

$\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k\}$  has at least two distinct points and no obtuse angles; (5)

$\{\mathbf{u}_{1,1}, \mathbf{u}_{2,2}, \dots, \mathbf{u}_{k,k}\}$  is an orthogonal set; (6)

If  $i \neq j$ , then  $\mathbf{u}_{i,j} = -\mathbf{u}_{j,i}$ ; (7)

If  $\mathbf{p}_i \neq \mathbf{p}_j$ , then  $\mathbf{u}_{i,j} = |\mathbf{p}_j \mathbf{p}_i|^{-1}(\mathbf{p}_j - \mathbf{p}_i)$ ; (8)

For any distinct  $i, j$ ,  $\mathbf{u}_{i,i}$  is orthogonal to  $\mathbf{u}_{i,j}$ ; (9)

Each  $\mathbf{u}_{i,i}$  is orthogonal to the subspace  $\text{lin}\{\mathbf{p}_j - \mathbf{p}_1 : j = 2, \dots, k\}$ ; (10)

If  $\mathbf{p}_i = \mathbf{p}_{i'} \neq \mathbf{p}_j$ , then  $\mathbf{u}_{i,i'}$  is orthogonal to  $\mathbf{u}_{i,j} = \mathbf{u}_{i',j}$ . (11)

*Proof.* The proof consists of three steps.

*Step 1.* We will use a geometric Ramsey-type result from [11] and the pigeon-hole principle to show that for any  $\varepsilon > 0$  there exists  $N$  such that for any  $K_k(N)$  with classes  $V_1, \dots, V_k$  contained in some double-normal graph in  $\mathbb{R}^d$ , there exist points  $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i \in V_i$  ( $i = 1, \dots, k$ ) such that

$$\angle \mathbf{a}_i \mathbf{b}_i \mathbf{c}_i > \pi - \varepsilon, \quad i = 1, \dots, k, \quad (12)$$

$$|\mathbf{a}_{i+1} \mathbf{c}_{i+1}| \leq \varepsilon |\mathbf{a}_i \mathbf{c}_i|, \quad i = 1, \dots, k-1, \quad (13)$$

$$|\mathbf{a}_i \mathbf{b}_i| \geq \frac{1}{2} |\mathbf{a}_i \mathbf{c}_i|, \quad i = 1, \dots, k. \quad (14)$$

*Step 2.* We use the results from §2 to show that if we set  $\mathbf{u}_{i,i} := |\mathbf{a}_i \mathbf{b}_i|^{-1}(\mathbf{a}_i - \mathbf{b}_i)$  and  $\mathbf{u}_{i,j} := |\mathbf{b}_j \mathbf{b}_i|^{-1}(\mathbf{b}_j - \mathbf{b}_i)$ , then

$$|\langle \mathbf{u}_{i,i}, \mathbf{u}_{i,j} \rangle| < \varepsilon, \quad i, j = 1, \dots, k, \quad i \neq j, \quad (15)$$

$$|\langle \mathbf{u}_{i,i}, \mathbf{u}_{j,j} \rangle| < 4\varepsilon, \quad i, j = 1, \dots, k, \quad i \neq j. \quad (16)$$

*Step 3.* The proposition will follow by setting  $\varepsilon := 1/n$  and taking subsequences of the sequences  $\mathbf{a}_i^{(n)}, \mathbf{b}_i^{(n)}, \mathbf{c}_i^{(n)}$ ,  $i = 1, \dots, k$ , such that  $\mathbf{b}_i^{(n)}$  converges to  $\mathbf{p}_i$ , and each  $\mathbf{u}_{i,j}^{(n)}$  converges, as  $n \rightarrow \infty$ . The details follow.

Let  $\varepsilon > 0$  be given. Write  $t := \lceil (\varepsilon \cos \varepsilon)^{-1} \rceil$ . In Step 1, applying [11, Theorem 4] we first choose a sufficiently large  $N$  depending only on  $\varepsilon$  and  $d$  such that each class  $V_i$  of any  $K_k(N)$  contained in a double-normal graph in  $\mathbb{R}^d$

**Algorithm 1:** Pruning the sets  $V_i$ **for**  $i = 1$  **to**  $k$  **do**(Note that here  $|V_j| = 2t^{k-i} + 1$  for all  $j \geq i$ )relabel  $V_i, \dots, V_k$  such that  $\text{diam}(V_i) = \max\{\text{diam}(V_j) : j > i\}$ **for**  $j = i + 1$  **to**  $k$  **do**find  $V'_j \subseteq V_j$  such that  $|V'_j| = 2t^{k-i-1} + 1$ and  $\text{diam}(V'_j) \leq \varepsilon \text{diam}(V_j)$ ;replace  $V_j$  by  $V'_j$ ;

has a subset  $V'_i$  of size  $2t^{k-1} + 1$  such that for any  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  from the same  $V'_i$  with  $\mathbf{a} \neq \mathbf{b}$  and  $\mathbf{c} \neq \mathbf{d}$ , the angle between the lines  $\mathbf{ab}$  and  $\mathbf{cd}$  is less than  $\varepsilon$ . We now replace the original  $V_i$  by  $V'_i$ . If we assume  $\varepsilon < \pi/3$ , we obtain a natural linear ordering (more precisely, a betweenness relation) on the points of each  $V_i$ , by defining for each  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V_i$  that  $\mathbf{y}$  is *between*  $\mathbf{x}$  and  $\mathbf{z}$  if  $\angle \mathbf{xyz} > \pi - \varepsilon$ . Then  $|\mathbf{yx}| < |\mathbf{zx}|$  whenever  $\mathbf{y}$  is between  $\mathbf{x}$  and  $\mathbf{z}$ .

Next we run Algorithm 1 on  $V_1, \dots, V_k$ . Note that at the start of the outer **for** loop,  $|V_j| = 2t^{k-i} + 1$  for all  $j = i, \dots, k$ . That we can find a  $V'_j$  as required inside the inner **for** loop, is seen as follows. Write  $V_j := \{\mathbf{p}_1, \dots, \mathbf{p}_{2t^{k-i}+1}\}$  with the points in their natural order (where  $\mathbf{p}_j$  is between  $\mathbf{p}_i$  and  $\mathbf{p}_k$  if  $\angle \mathbf{p}_i \mathbf{p}_j \mathbf{p}_k > \pi - \varepsilon$ ). Let  $\mathbf{p}'_i$  be the orthogonal projection of  $\mathbf{p}_i$  onto the line  $\ell$  through  $\mathbf{p}_1$  and  $\mathbf{p}_{2t^{k-i}+1}$ . Since  $\varepsilon < \pi/2$ , the points  $\mathbf{p}'_i$  are in order on  $\ell$ , and

$$\begin{aligned} |\mathbf{p}_1 \mathbf{p}_{2t^{k-i}+1}| &= |\mathbf{p}'_1 \mathbf{p}'_{2t^{k-i}+1}| \\ &= \sum_{s=1}^t |\mathbf{p}'_{2t^{k-i-1}(s-1)+1} \mathbf{p}'_{2t^{k-i-1}s+1}| \\ &> \cos \varepsilon \sum_{s=1}^t |\mathbf{p}_{2t^{k-i-1}(s-1)+1} \mathbf{p}_{2t^{k-i-1}s+1}|, \end{aligned}$$

where the last inequality holds, because the angle between  $\ell$  and the line through any two  $\mathbf{p}_i$  is less than  $\varepsilon$ . Thus,

$$\frac{1}{t} \sum_{s=1}^t |\mathbf{p}_{2t^{k-i-1}(s-1)+1} \mathbf{p}_{2t^{k-i-1}s+1}| < \frac{1}{t \cos \varepsilon} |\mathbf{p}_1 \mathbf{p}_{2t^{k-i}+1}| < \varepsilon |\mathbf{p}_1 \mathbf{p}_{2t^{k-i}+1}|.$$

It follows that for some  $s \in \{1, \dots, t\}$ ,

$$|\mathbf{p}_{2t^{k-i-1}(s-1)+1} \mathbf{p}_{2t^{k-i-1}s+1}| < \varepsilon |\mathbf{p}_1 \mathbf{p}_{2t^{k-i}+1}|.$$

Let  $V'_j := \{\mathbf{p}_{2t^{k-i-1}(s-1)+1}, \dots, \mathbf{p}_{2t^{k-i-1}s+1}\}$ . Then  $|V'_j| = 2t^{k-i-1} + 1$  and

$$\text{diam}(V'_j) < \varepsilon |\mathbf{p}_1 \mathbf{p}_{2t^{k-i}+1}| = \varepsilon \text{diam}(V_j).$$

When the algorithm is done, we have sets  $V_1, \dots, V_k$  such that  $\text{diam}(V_{i+1}) \geq \varepsilon \text{diam}(V_i)$  for each  $i = 1, \dots, k-1$ , and  $|V_i| = 2t^{k-i} + 1 \geq 3$  for each  $i = 1, \dots, k$ . Let  $\mathbf{a}_i \mathbf{c}_i$  be a diameter of  $V_i$  and choose any  $\mathbf{b}_i \in V_i \setminus \{\mathbf{a}_i, \mathbf{c}_i\}$ . Then (12) and (13) hold. To ensure (14), exchange  $\mathbf{a}_i$  and  $\mathbf{c}_i$  if necessary such that  $|\mathbf{a}_i \mathbf{b}_i| \geq |\mathbf{c}_i \mathbf{b}_i|$ . Then (14) follows from the triangle inequality.

In Step 2 we show (15) and (16). Let  $1 \leq i, j \leq k$ ,  $i \neq j$ . Without loss of generality,  $i < j$ . Then (15) follows upon applying Lemma 4 with  $\mathbf{x} := \mathbf{b}_i$ ,  $\mathbf{y}_1 := \mathbf{a}_j$ ,  $\mathbf{y}_2 := \mathbf{b}_j$ ,  $\mathbf{y}_3 := \mathbf{c}_j$ .

If  $\angle \mathbf{a}_i \mathbf{b}_j \mathbf{b}_i \geq \pi/6$ , then by Lemma 5 with  $\mathbf{x}_1 := \mathbf{a}_i$ ,  $\mathbf{x}_2 := \mathbf{b}_i$ ,  $\mathbf{y}_1 := \mathbf{a}_j$ ,  $\mathbf{y}_2 := \mathbf{b}_j$ ,  $\mathbf{y}_3 := \mathbf{c}_j$ ,

$$|\langle \mathbf{u}_{i,i}, \mathbf{u}_{j,j} \rangle| < \frac{2\varepsilon}{\sin \angle \mathbf{a}_i \mathbf{b}_j \mathbf{b}_i} \leq \frac{2\varepsilon}{\sin \pi/6} = 4\varepsilon.$$

If  $\angle \mathbf{a}_i \mathbf{b}_j \mathbf{b}_i < \pi/6$ , then by Lemma 6 with  $\mathbf{x}_1 := \mathbf{a}_i$ ,  $\mathbf{x}_2 := \mathbf{b}_i$ ,  $\mathbf{y}_1 := \mathbf{a}_j$ ,  $\mathbf{y}_2 := \mathbf{b}_j$ ,

$$\begin{aligned} |\langle \mathbf{u}_{i,i}, \mathbf{u}_{j,j} \rangle| &< \frac{\sqrt{2}}{\cos^2 \angle \mathbf{a}_i \mathbf{b}_j \mathbf{b}_i} \frac{|\mathbf{a}_j \mathbf{b}_j|}{|\mathbf{a}_i \mathbf{b}_i|} \\ &< \frac{\sqrt{2}}{\cos^2(\pi/6)} \frac{|\mathbf{a}_j \mathbf{c}_j|}{\frac{1}{2} |\mathbf{a}_i \mathbf{c}_i|} < (8\sqrt{2}/3)\varepsilon < 4\varepsilon, \end{aligned}$$

which shows (16).

In Step 3, we let  $n \in \mathbb{N}$  be arbitrary, set  $\varepsilon := 1/n$ , and choose  $\mathbf{a}_i^{(n)}, \mathbf{b}_i^{(n)}, \mathbf{c}_i^{(n)}$ ,  $i = 1, \dots, k$ , as in the first stage of the proof. We may assume, after translating and scaling each  $\bigcup_{i=1}^k V_i^{(n)}$  if necessary, that  $\{\mathbf{b}_1^{(n)}, \dots, \mathbf{b}_k^{(n)}\}$  has diameter 1 and is contained in the unit ball. Thus, we may pass to subsequences to assume that for each  $i$ ,  $\mathbf{b}_i^{(n)}$  converges to  $\mathbf{p}_i$ , say,

$$\mathbf{u}_{i,i}^{(n)} := |\mathbf{a}_i^{(n)} \mathbf{b}_i^{(n)}|^{-1} (\mathbf{a}_i^{(n)} - \mathbf{b}_i^{(n)})$$

converges to  $\mathbf{u}_{i,i}$ , say, and

$$\mathbf{u}_{i,j}^{(n)} := |\mathbf{b}_j^{(n)} \mathbf{b}_i^{(n)}|^{-1} (\mathbf{b}_j^{(n)} - \mathbf{b}_i^{(n)})$$

converges to  $\mathbf{u}_{i,j}$ , say. Then  $\text{diam}\{\mathbf{p}_1, \dots, \mathbf{p}_k\} = 1$ , and since there are no obtuse angles in  $\{\mathbf{b}_1^{(n)}, \dots, \mathbf{b}_k^{(n)}\}$ , there will still be no obtuse angles between distinct elements of  $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ . Thus, (5) holds. Also, (6) follows from (16), (7) from the definition of  $\mathbf{u}_{i,j}^{(n)}$ , (8) from the definitions of  $\mathbf{u}_{i,i}^{(n)}$  and  $\mathbf{p}_i$ , and (9) from (15). Properties (8) and (9) immediately imply that  $\mathbf{u}_{i,i}$  is orthogonal to  $\mathbf{p}_i - \mathbf{p}_j$  for all  $j \neq i$ . Since the subspace  $\text{lin}\{\mathbf{p}_i - \mathbf{p}_j : j \neq i\}$  is the same for all  $i$ , we obtain (10).

Finally, suppose  $\mathbf{p}_i = \mathbf{p}_{i'} \neq \mathbf{p}_j$ . Since  $\angle \mathbf{b}_i^{(n)} \mathbf{b}_j^{(n)} \mathbf{b}_{i'}^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$  and  $\angle \mathbf{b}_i \mathbf{b}_{i'} \mathbf{b}_j$  is not obtuse, we obtain that  $\angle \mathbf{b}_i^{(n)} \mathbf{b}_{i'}^{(n)} \mathbf{b}_j^{(n)} \rightarrow \pi/2$  and  $\angle \mathbf{b}_{i'}^{(n)} \mathbf{b}_i^{(n)} \mathbf{b}_j^{(n)} \rightarrow \pi/2$  as  $n \rightarrow \infty$ , giving  $\mathbf{u}_{i,i'} \perp \mathbf{u}_{i,j}$ . This shows (11).  $\square$



*Proof of Theorem 7.* Let  $k = k(d)$ . Consider the points  $\mathbf{p}_1, \dots, \mathbf{p}_k$  and vectors  $\mathbf{u}_{i,j}$ ,  $1 \leq i, j \leq k$ , given by Proposition 8. There exist distinct  $i$  and  $j$  such that  $\mathbf{p}_i \neq \mathbf{p}_j$ . By (6), the  $k$  unit vectors  $\mathbf{u}_{1,1}, \dots, \mathbf{u}_{k,k}$  are pairwise orthogonal. By (10), they are also orthogonal to  $\mathbf{p}_i - \mathbf{p}_j$ , which is a multiple of  $\mathbf{u}_{i,j}$  by (8). Thus, we have found  $k + 1$  pairwise orthogonal vectors. That is,  $k(d) + 1 \leq d$ .  $\square$

#### §4. Constructions with many strict double-normal pairs.

**THEOREM 9.** *Let  $m \geq 2$ . Suppose that there exist  $m$  points  $\mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{R}^d$  and  $m$  unit vectors  $\mathbf{u}_1, \dots, \mathbf{u}_m \in \mathbb{R}^d$  such that, for all triples of distinct  $i, j, k$ , the angle  $\angle \mathbf{p}_i \mathbf{p}_j \mathbf{p}_k$  is acute, and*

$$\langle \mathbf{u}_i, \mathbf{p}_i - \mathbf{p}_j \rangle < \langle \mathbf{u}_i, \mathbf{p}_k - \mathbf{p}_j \rangle < \langle \mathbf{u}_i, \mathbf{p}_j - \mathbf{p}_i \rangle. \quad (17)$$

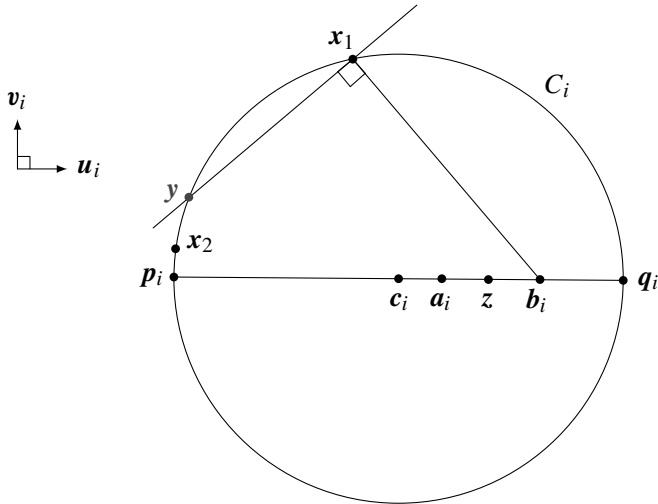
*Then, for any  $N \in \mathbb{N}$ , there exists a strict double-normal graph in  $\mathbb{R}^{d+m}$  containing a complete  $m$ -partite  $K_m(N)$ . In particular,  $k'(d+m) \geq m$ .*

Geometrically, (17) means that if we project the points  $\mathbf{p}_1, \dots, \mathbf{p}_m$  orthogonally onto the line through  $\mathbf{p}_i$  parallel to  $\mathbf{u}_i$ , then the projected points are on the ray from  $\mathbf{p}_i$  in the direction of  $\mathbf{u}_i$ , and the furthest is at less than twice the distance from  $\mathbf{p}_i$  than the closest (other than  $\mathbf{p}_i$ ).

*Proof.* Identify  $\mathbb{R}^d$  with the first  $d$  coordinates of  $\mathbb{R}^{d+m}$ , and let  $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^{d+m}$  be pairwise orthogonal unit vectors that are also orthogonal to  $\mathbb{R}^d$ . We will construct countably infinite sets  $V_1, \dots, V_m \subset \mathbb{R}^{d+m}$ , with each  $V_i$  on a circular arc through  $\mathbf{p}_i$  in the plane  $\Pi_i := \mathbf{p}_i + \text{lin}\{\mathbf{u}_i, \mathbf{v}_i\}$ . Then we will verify that for any distinct  $i, j$  and any  $\mathbf{x} \in V_i$  and  $\mathbf{y} \in V_j$ ,  $\mathbf{x}\mathbf{y}$  is a strict double-normal pair of  $\bigcup_i V_i$ .

We will use a small  $\varepsilon > 0$  that will depend only on the given points  $\mathbf{p}_1, \dots, \mathbf{p}_m$  and vectors  $\mathbf{u}_1, \dots, \mathbf{u}_m$ . As the proof progresses, we will put finitely many constraints on  $\varepsilon$ , all depending only on the points  $\mathbf{p}_i$  and vectors  $\mathbf{u}_i$ .

Let  $\alpha_i := \min_{j \neq i} \langle \mathbf{u}_i, \mathbf{p}_j \rangle$  and  $\beta_i := \max_j \langle \mathbf{u}_i, \mathbf{p}_j \rangle$ . By condition (17),  $\langle \mathbf{u}_i, \mathbf{p}_i \rangle - \alpha_i < \beta_i - \alpha_i < \alpha_i - \langle \mathbf{u}_i, \mathbf{p}_i \rangle$ , hence  $\langle \mathbf{u}_i, \mathbf{p}_i \rangle < \frac{1}{2}(\beta_i + \langle \mathbf{u}_i, \mathbf{p}_i \rangle) < \alpha_i$ . We choose  $\varepsilon > 0$  small enough so that  $\frac{1}{2}(\beta_i + \varepsilon + \langle \mathbf{u}_i, \mathbf{p}_i \rangle) < \alpha_i - \varepsilon$  for all  $i$ . Choose any  $r_i \in (\frac{1}{2}(\beta_i + \varepsilon + \langle \mathbf{u}_i, \mathbf{p}_i \rangle), \alpha_i - \varepsilon)$ , and set  $\mathbf{c}_i := \mathbf{p}_i + r_i \mathbf{u}_i$ ,  $\mathbf{a}_i := \mathbf{p}_i + (\alpha_i - \varepsilon) \mathbf{u}_i$ ,  $\mathbf{b}_i := \mathbf{p}_i + (\beta_i + \varepsilon) \mathbf{u}_i$ ,  $\mathbf{q}_i := \mathbf{p}_i + 2r_i \mathbf{u}_i$  (Figure 1). Denote the circle with centre  $\mathbf{c}_i$  and radius  $r_i$  in the plane  $\Pi_i$  by  $C_i$ . Then  $\mathbf{p}_i \mathbf{q}_i$  is a diameter of  $C_i$  parallel to  $\mathbf{u}_i$ , and  $\mathbf{a}_i$  and  $\mathbf{b}_i$  are strictly between  $\mathbf{c}_i$  and  $\mathbf{q}_i$ . Choose any  $\mathbf{x}_1 \in C_i \setminus \{\mathbf{p}_i\}$  such that  $\angle \mathbf{x}_1 \mathbf{c}_i \mathbf{p}_i$  is acute. We will now recursively choose  $\mathbf{x}_2, \mathbf{x}_3, \dots$  on the minor arc  $\gamma_i$  of  $C_i$  between  $\mathbf{x}_1$  and  $\mathbf{p}_i$  such that for any  $\mathbf{z}$  on the segment  $\mathbf{a}_i \mathbf{b}_i$ , the angle  $\angle \mathbf{z} \mathbf{x}_t \mathbf{x}_s$  is acute for all distinct  $s, t \in \mathbb{N}$ . Assume that for some  $t \in \mathbb{N}$  we have already chosen  $\mathbf{x}_1, \dots, \mathbf{x}_t \in \gamma_i$  with  $\mathbf{x}_{s+1}$  between  $\mathbf{x}_s$  and  $\mathbf{p}_i$  for each  $s = 1, \dots, t-1$ , and such that  $\angle \mathbf{z} \mathbf{x}_j \mathbf{x}_k$  is acute for all  $1 \leq j, k \leq t$ ,  $j \neq k$ , and for all  $\mathbf{z}$  on the segment  $\mathbf{a}_i \mathbf{b}_i$ . Since  $\mathbf{p}_i \mathbf{x}_t \mathbf{q}_i$  is a right angle,  $\angle \mathbf{p}_i \mathbf{x}_t \mathbf{b}_i$  is acute, and the line in  $\Pi_i$  through  $\mathbf{x}_t$  and perpendicular to  $\mathbf{b}_i \mathbf{x}_t$  intersects  $C_i$  in a point  $\mathbf{y} \in \gamma_i$  between  $\mathbf{x}_t$  and  $\mathbf{p}_i$ . Let  $\mathbf{x}_{t+1}$  be any point on  $\gamma_i$

Figure 1: Constructing  $V_i = \{x_t : t \in \mathbb{N}\}$ .

between  $y$  and  $p_i$ . Now consider any  $z$  on the segment  $a_i b_i$ . We have to show that  $\angle z x_{t+1} x_s$  and  $\angle z x_s x_{t+1}$  are acute for all  $s = 1, \dots, t$ . This can be simply seen as follows:

$$\angle z x_{t+1} x_s \leq \angle z x_{t+1} x_t \leq \angle c_i x_{t+1} x_t < \pi/2$$

and

$$\angle z x_s x_{t+1} \leq \angle z x_t x_{t+1} \leq \angle b_i x_t x_{t+1} < \angle b_i x_t y = \pi/2.$$

Finally, let  $V_i := \{x_t : t \in \mathbb{N}\}$ . Then  $\text{diam } V_i = |p_i x_1|$ , which can be made arbitrarily small by choosing  $x_1$  close enough to  $p_i$ . We can assume that all  $\text{diam}(V_i) < \varepsilon$ . This finishes the construction.

Let  $1 \leq i < j \leq m$ ,  $x \in V_i$  and  $y \in V_j$ . We have to show that all  $z \in \bigcup_i V_i \setminus \{x, y\}$  are in the open slab bounded by the hyperplanes through  $x$  and  $y$  orthogonal to  $xy$ . First consider the case where  $z \in V_k$ ,  $k \neq i, j$ . Since  $\angle p_i p_j p_k$  and  $\angle p_j p_i p_k$  are acute,  $\langle p_i - p_j, p_k - p_j \rangle > 0$  and  $\langle p_j - p_i, p_k - p_i \rangle > 0$ . Noting that  $|xp_i|, |yp_j|, |zp_k| < \varepsilon$ , it follows that  $\langle x - y, z - y \rangle > 0$  and  $\langle y - x, z - x \rangle > 0$  if  $\varepsilon$  is sufficiently small, depending only on the given points. That is,  $z$  is in the open slab determined by  $xy$ .

Next consider the case where  $z \in V_i \cup V_j$ . Without loss of generality,  $z \in V_i$ . Then

$$\langle x - y, z - y \rangle = \langle x - y, z - x \rangle + |xy|^2 \geq -\varepsilon|xy| + |xy|^2 > 0,$$

as long as  $\varepsilon < |xy|$ . It remains to verify that  $\langle y - x, z - x \rangle > 0$ . Denote the orthogonal projection of a point  $p \in \mathbb{R}^{d+m}$  onto the plane  $\Pi_i$  by  $p'$ . Since  $V_j \subset \Pi_j \subseteq \mathbb{R}^d + \text{lin}\{v_j\}$ , it follows that  $p'_j, y' \in p_i + \text{lin}\{u_i\}$ . In particular,  $p'_j$  is also the orthogonal projection of  $p_j$  onto the line  $p_i + \text{lin}\{u_i\}$ . By hypothesis,

$\mathbf{p}'_j = \mathbf{p}_i + \lambda \mathbf{u}_i$  for some  $\lambda \in [\alpha_i, \beta_i]$ . Since  $|\mathbf{p}'_j \mathbf{y}'| \leq |\mathbf{p}_j \mathbf{y}| < \varepsilon$ , it follows that  $\mathbf{y}' = \mathbf{p}_i + \mu \mathbf{u}_i$  where  $\mu \in [\alpha_i - \varepsilon, \beta_i + \varepsilon]$ , that is,  $\mathbf{y}'$  is on the segment  $\mathbf{a}_i \mathbf{b}_i$ . By construction, the angle  $\angle \mathbf{y}' \mathbf{x} \mathbf{z}$  is acute, hence  $\langle \mathbf{y} - \mathbf{x}, \mathbf{z} - \mathbf{x} \rangle = \langle \mathbf{y}' - \mathbf{x}, \mathbf{z} - \mathbf{x} \rangle > 0$ .  $\square$

COROLLARY 10. We have  $k'(d) \geq \lceil d/2 \rceil$ .

*Proof.* Let  $m := \lceil d/2 \rceil$ . Let  $\mathbf{p}_1, \dots, \mathbf{p}_m$  be the vertices of a regular simplex in  $\mathbb{R}^{m-1}$  inscribed in the unit sphere. Then  $\mathbf{p}_i$  and  $\mathbf{u}_i := -\mathbf{p}_i$  satisfy the conditions of Theorem 9. It follows that  $k'(d) \geq k'(2m-1) \geq m$ .  $\square$

THEOREM 11. There exist  $m := \lfloor \frac{1}{4} e^{d/20} \rfloor$  distinct points  $\mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{R}^d$  and unit vectors  $\mathbf{u}_1, \dots, \mathbf{u}_m \in \mathbb{R}^d$  such that for all distinct  $1 \leq i, j, k \leq m$ , the angle  $\angle \mathbf{p}_i \mathbf{p}_j \mathbf{p}_k$  is acute, and condition (17) is satisfied.

The proof of Theorem 11 is probabilistic, and is a modification of an argument of Erdős and Füredi [2]. Write  $[d]$  for the set  $\{1, 2, \dots, d\}$  of all integers from 1 to  $d$ . For any  $A \subseteq [d]$ , let  $\chi(A) \in \{0, 1\}^d$  denote its characteristic vector. The routine proofs of the following three lemmas are omitted.

LEMMA 12 [2, Lemma 2.3]. Let  $A, B$ , and  $C$  be distinct subsets of  $[d]$ . Then we have  $\angle \chi(A) \chi(C) \chi(B) \leq \pi/2$ , and equality holds if and only if  $A \cap B \subseteq C \subseteq A \cup B$ .

LEMMA 13 [2]. If  $A, B$ , and  $C$  are subsets of  $[d]$  chosen independently and uniformly, then we have  $\Pr[A \cap B \subseteq C \subseteq A \cup B] = (3/4)^d$ .

LEMMA 14. Let  $A, B, C \subseteq [d]$  and consider the unit vector

$$\mathbf{u} := (1/\sqrt{d})(\chi([d]) - 2\chi(A)).$$

Then we have  $\langle \mathbf{u}, \chi(A) \rangle \leq \langle \mathbf{u}, \chi(B) \rangle$ , with equality if and only if  $A = B$ . Also,

$$\langle \mathbf{u}, \chi(B) - \chi(C) \rangle \geq \langle \mathbf{u}, \chi(C) - \chi(A) \rangle$$

if and only if

$$4|A \cap C| + |B| \geq 2|A \cap B| + |A| + 2|C|.$$

LEMMA 15. If  $A, B$ , and  $C$  are subsets of  $[d]$  chosen independently and uniformly, then we have

$$\Pr[4|A \cap C| + |B| \geq 2|A \cap B| + |A| + 2|C|] \leq \left(\frac{65}{72}\right)^d < e^{-d/10}.$$

*Proof.* Let  $X$  be the random variable

$$X := 4|A \cap C| + |B| - 2|A \cap B| - |A| - 2|C| = \sum_{i=1}^d X_i,$$

where  $X_i$  is the contribution of the element  $i \in [d]$  to  $X$ , that is,

$$X_i := \begin{cases} 1 & \text{if } i \in B \setminus (A \cup C) \text{ or } i \in (A \cap C) \setminus B, \\ 0 & \text{if } i \in A \cap B \cap C \text{ or } i \notin A \cup B \cup C, \\ -1 & \text{if } i \in A \setminus (B \cup C) \text{ or } i \in (B \cap C) \setminus A, \\ -2 & \text{if } i \in C \setminus (A \cup B) \text{ or } i \in (A \cap B) \setminus C. \end{cases}$$

Note that

$$\Pr[X_i = 1] = \Pr[X_i = 0] = \Pr[X_i = -1] = \Pr[X_i = -2] = 1/4.$$

We now bound  $\Pr[X \geq 0]$  from above. For any  $\lambda \geq 1$ ,

$$\begin{aligned} \Pr[X \geq 0] &= \Pr[\lambda^X \geq 1] \\ &\leq \mathbb{E}[\lambda^X] = \prod_{i=1}^d \mathbb{E}[\lambda^{X_i}] = \left( \frac{\lambda + 1 + \lambda^{-1} + \lambda^{-2}}{4} \right)^d, \end{aligned}$$

where we used Markov's inequality and independence. Set  $\lambda := 3/2$ , which is close to minimizing the right-hand side. This gives  $\Pr[X \geq 0] \leq (65/72)^d$ .  $\square$

*Proof of Theorem 11.* Let  $m := \lfloor (1/4)e^{d/20} \rfloor$ . Choose subsets  $A_1, \dots, A_{2m}$  randomly and independently from the set  $[d]$ . For  $i \in [d]$ , define  $\mathbf{p}_i := \chi(A_i)$  and  $\mathbf{u}_i := (1/\sqrt{d})(\chi([d]) - 2\chi(A_i))$ . Let  $i, j, k \in [d]$  be distinct.

Assume that  $A_i, A_j, A_k$  are distinct sets. Then by Lemma 12,  $\angle \mathbf{p}_i \mathbf{p}_k \mathbf{p}_j$  fails to be acute if and only if

$$A_i \cap A_j \subseteq A_k \subseteq A_i \cup A_j, \quad (18)$$

and condition (17) is violated if and only if

$$\langle \mathbf{u}_i, \chi(A_i) - \chi(A_j) \rangle \geq \langle \mathbf{u}_i, \chi(A_k) - \chi(A_j) \rangle \quad (19)$$

or

$$\langle \mathbf{u}_i, \chi(A_k) - \chi(A_j) \rangle \geq \langle \mathbf{u}_i, \chi(A_j) - \chi(A_i) \rangle. \quad (20)$$

Condition (19) is equivalent to  $\langle \mathbf{u}_i, \chi(A_i) \rangle \geq \langle \mathbf{u}_i, \chi(A_k) \rangle$ . This, in turn, is equivalent to  $A_i = A_k$ , by the first statement of Lemma 14, contradicting our assumption that  $A_i, A_j, A_k$  are distinct. By the second statement of Lemma 14, (20) is equivalent to

$$4|A_i \cap A_j| + |A_k| \geq 2|A_i \cap A_k| + |A_i| + 2|A_j|. \quad (21)$$

Thus, for distinct points  $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ , at least one of the conditions (18) and (21) holds if and only if  $\angle \mathbf{p}_i \mathbf{p}_k \mathbf{p}_j$  is a right angle or condition (17) is violated.

Note that if some two of the sets coincide, say  $A_i = A_k$ , then (18) also holds. Let us call a triple of distinct numbers  $(i, j, k)$  *bad* if at least one of (18) and (21) holds. It follows that if no triple  $(i, j, k)$  is bad, then all points  $\mathbf{p}_i$  are distinct,

all angles  $\angle p_i p_j p_k$  are acute, and condition (17) is also satisfied. We will show that with positive probability, some  $m$  of the  $A_1, \dots, A_{2m}$  will be without bad triples, which will prove the theorem.

By Lemmas 13 and 15 and the union bound, we obtain that

$$\Pr[(i, j, k) \text{ is bad}] \leq (3/4)^d + e^{-d/10} < 2e^{-d/10}.$$

By linearity of expectation, the expected number of bad triples is at most

$$2m(2m-1)(2m-2)2e^{-d/10} < 16m^3e^{-d/10}.$$

In particular, there exists a choice of subsets  $A_1, \dots, A_{2m} \subseteq [d]$  with less than  $16m^3e^{-d/10}$  bad triples. For each bad triple  $(i, j, k)$ , remove  $A_i$  from  $\{A_1, \dots, A_{2m}\}$ . We are left with more than  $2m - 16m^3e^{-d/10}$  sets without any bad triple. Since  $m \leq (1/4)e^{d/20}$  implies that  $2m - 16m^3e^{-d/10} \geq m$ , we obtain  $m$  points  $p_i$  with unit vectors  $u_i$  satisfying the theorem.  $\square$

**COROLLARY 16.** *We have  $k'(d) \geq d - O(\log d)$ .*

*Proof.* Let  $n$  be the unique integer such that

$$\lfloor (1/4)e^{n/20} \rfloor + n \leq d < \lfloor (1/4)e^{(n+1)/20} \rfloor + n + 1.$$

By Theorems 11 and 9,  $k'(m+n+1) \geq m$  for any  $m = 2, \dots, \lfloor (1/4)e^{(n+1)/20} \rfloor$ . In particular, if we take  $m = d - n - 1$ , we obtain

$$k'(d) \geq d - n - 1 > d - 20 \log(4d) - 1. \quad \square$$

*Acknowledgements.* We thank Endre Makai for a careful reading of the manuscript and for many enlightening comments. We also thank the referee for remarks that led to an improved manuscript.

### References

1. P. Erdős, On sets of distances of  $n$  points in Euclidean space. *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **5** (1960), 165–169.
2. P. Erdős and Z. Füredi, The greatest angle among  $n$  points in the  $d$ -dimensional Euclidean space. (*North-Holland Mathematics Studies* **75**), North-Holland (Amsterdam, 1983), 275–283.
3. P. Erdős and A. H. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc. (N.S.)* **52** (1946), 1087–1091.
4. B. Grünbaum, A proof of Vázsonyi's conjecture. *Bull. Res. Council Israel, Section A* **6** (1956), 77–78.
5. A. Heppes, Beweis einer Vermutung von A. Vázsonyi. *Acta Math. Acad. Sci. Hungar.* **7** (1956), 463–466.
6. A. Kupavskii, Diameter graphs in  $\mathbb{R}^d$ . *Discrete Comput. Geom.* **51** (2014), 842–858.
7. E. Makai Jr. and H. Martini, On the number of antipodal or strictly antipodal pairs of points in finite subsets of  $\mathbb{R}^d$ . In *Applied Geometry and Discrete Mathematics (DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **4**), American Mathematical Society (Providence, RI, 1991), 457–470.
8. E. Makai Jr. and H. Martini, On the number of antipodal or strictly antipodal pairs of points in finite subsets of  $\mathbb{R}^d$ . II. *Period. Math. Hungar.* **27** (1993), 185–198.

9. E. Makai Jr., H. Martini, H. Nguên, V. Soltan and I. Talata, On the number of antipodal or strictly antipodal pairs of points in finite subsets of  $\mathbb{R}^d$ . III. Manuscript.
10. H. Martini and V. Soltan, Antipodality properties of finite sets in Euclidean space. *Discrete Math.* **290** (2005), 221–228.
11. J. Pach, A remark on transversal numbers. In *The Mathematics of Paul Erdős II (Algorithms and Combinatorics 14)* (ed. R. L. Graham *et al*), Springer (Berlin, 1997), 310–317.
12. J. Pach and K. J. Swanepoel, Double-normal pairs in the plane and on the sphere. *Beitr. Algebra Geom.*, to appear, [arXiv:1404.2624](https://arxiv.org/abs/1404.2624).
13. S. Straszewicz, Sur un problème géométrique de P. Erdős. *Bull. Acad. Polon. Sci. Cl. III.* **5** (1957), 39–40, IV–V.
14. K. J. Swanepoel, Unit distances and diameters in Euclidean spaces. *Discrete Comput. Geom.* **41** (2009), 1–27.

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